

1D Coulomb problem with deformed Heisenberg algebra

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In this report we consider one dimensional quantum mechanics with the following deformation

$$[X, P] = i(1 + \beta P^2). \quad (1)$$

Such a deformation implies that there exists minimal resolution of length $\Delta X \geq \sqrt{\beta}$, i.e., there is no possibility to measure coordinate X with accuracy more than ΔX . It is expected that deformation can help to avoid divergences.

We solve eigenvalue equation for Coulomb-type potential

$$P^2\psi - \frac{\alpha}{X}\psi = E\psi. \quad (2)$$

Momentum representation

$$P = p, \quad X = i(1 + \beta p^2) \frac{d}{dp}. \quad (3)$$

We have to redefine scalar product to ensure Hermiticity of X :

$$\langle \phi | \psi \rangle = \int_{-\infty}^{\infty} \frac{\phi^*(p) \psi(p)}{1 + \beta p^2} dp, \quad \langle \phi | A \psi \rangle = \int_{-\infty}^{\infty} \frac{\phi^*(p)}{1 + \beta p^2} A \psi(p) dp. \quad (4)$$

It is naturally to demand

$$X \frac{1}{X} = 1 \quad (5)$$

Multiplication of eigenvalue equation gives

$$XP^2\psi - \alpha\psi = EX\psi. \quad (6)$$

Solution of this equation reads

$$\psi_\epsilon(p) = \frac{C_\epsilon}{\epsilon + p^2} e^{\left(\frac{i\alpha}{1-\epsilon\beta} \left(\frac{1}{\sqrt{\epsilon}} \arctan \frac{p}{\sqrt{\epsilon}} - \sqrt{\beta} \arctan \sqrt{\beta} p \right) \right)}, \quad (7)$$

where $\epsilon = -E$.

Properties of eigenfunctions:

- if $\epsilon > 0$ then $\langle \psi_\epsilon | \psi_\epsilon \rangle$ exists.

Spectrum

Integrating of equation (6) gives

$$p^2\psi(p) + i\alpha \int_{-\infty}^p \frac{\psi(q) dq}{1 + \beta q^2} + c[\psi] = E\psi(p). \quad (8)$$

It looks like as an eigenvalue equation. Thus,

$$\frac{1}{X}\psi(p) = -i \int_{-\infty}^p \frac{\psi(q) dq}{1 + \beta q^2} - \frac{c[\psi]}{\alpha}. \quad (9)$$

Substituting eigenfunction into (8) we obtain

$$c[\psi_\epsilon] = - \lim_{p \rightarrow -\infty} (p^2 + \epsilon)\psi_\epsilon(p) = -C_\epsilon e^{\frac{-i\alpha\pi}{2(\sqrt{\epsilon} + \sqrt{\beta\epsilon})}}. \quad (10)$$

We require that $\frac{1}{X}$ to be an Hermitian operator on the set of eigenfunctions:

$$\left\langle \frac{1}{X} \psi_{\epsilon_1} \middle| \psi_{\epsilon_2} \right\rangle = \left\langle \psi_{\epsilon_1} \middle| \frac{1}{X} \psi_{\epsilon_2} \right\rangle. \quad (11)$$

As a result

$$\sin[g(\epsilon_1) - g(\epsilon_2)] = 0, \quad (12)$$

where

$$g(\epsilon) = \frac{\alpha\pi}{2(\sqrt{\epsilon} + \sqrt{\beta\epsilon})}.$$

Fixing one eigenvalue we obtain condition on the rest

$$\frac{\alpha}{2(\sqrt{\epsilon} + \sqrt{\beta\epsilon})} = \delta_0 + n, \quad (13)$$

Let us analyze case of $\delta_0 = 0$ more thoroughly. Quantization condition reads

$$\frac{\alpha}{2(\sqrt{\epsilon} + \sqrt{\beta\epsilon})} = n, \quad (14)$$

$$E_n = -\epsilon = -\frac{\left(1 - \sqrt{1 + \frac{2\alpha}{n}\sqrt{\beta}}\right)^2}{4\beta} \simeq -\frac{\alpha^2}{4n^2} + \frac{\alpha^3}{4n^3}\sqrt{\beta} - \frac{5\alpha^4}{16n^4}\beta, \quad (15)$$

where $n = 1, 2, \dots$

In general $\delta_0 \neq 0$ and

$$E_n = -\frac{\left(1 - \sqrt{1 + \frac{2\alpha}{n+\delta_0}\sqrt{\beta}}\right)^2}{4\beta}, \quad n = 0, 1, 2, \dots \quad (16)$$

Results

- The first correction to the spectrum is proportional to $\sqrt{\beta}$
- There exist several spectral families. It means that deformation does not help to avoid singularity.